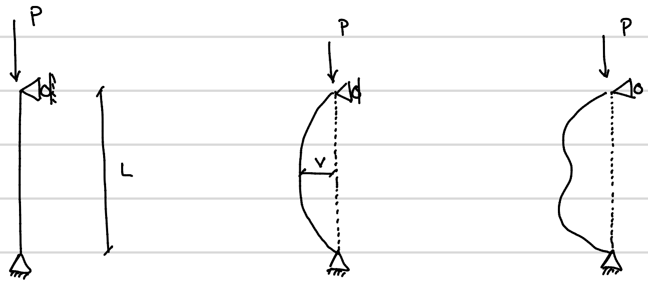
March 10, 2026
Column Buckling
Derivation

Column Buckling (instability)

$e^{i\pi} + 1 = 0$

* $e^{i\theta} = i\sin\theta + \cos\theta$ *

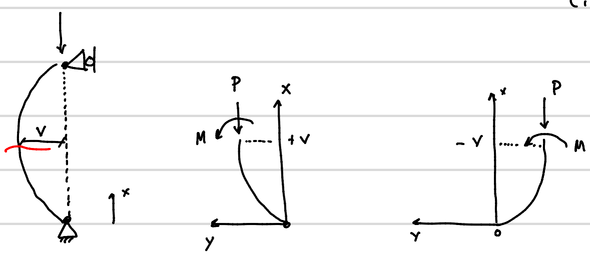
Leonard Euler c.a. 1757



$P < P_{cr}$ — stable equilibrium

$P = P_{cr}$ — Neutral Equilibrium

$P > P_{cr}$ — Unstable Equilibrium (i.e., buckling)



$\Sigma M_o = 0$ (origin)

$M + P(v) = 0$ $\qquad$ $M - P(-v) = 0$

$M + Pv = 0$ $\qquad$ $M + Pv = 0$

$EIv'' = \underline{M(x)}$

* $\underline{EI}\, v'' + \underline{P}v = \underline{0}$ — homogeneous, 2nd order linear differential equation with constant coefficients

let $k^2 = \frac{P}{EI}$

$v'' + k^2 v = 0$ — general solution $v = C_1 \sin(kx) + C_2 \cos(kx)$

$k^2 = \frac{P}{EI}$

C_1, C_2 constants of integration (use boundary conditions)

1st B.C. $v(0) = 0$ $\quad \therefore \quad C_2 = 0$ $\qquad$ $v = C_1 \sin(kx)$

2nd B.C. $v(L) = 0$ $\quad \therefore \quad C_1 \sin(kL) = 0 \longrightarrow$ two possibilities

Case 1

$C_1 = 0$, trivial solution

Case 2

$\sin(kL) = 0$, * buckling solution * $\qquad kL = 0, \pi, 2\pi, \ldots$

consider $kL = n\pi$ $\quad n = 1, 2, 3, \ldots$

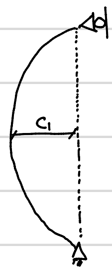
$P = k^2 EI$ $\quad \therefore \quad P = \dfrac{n^2 \pi^2 EI}{L^2}$ $\quad$ P_{cr} for simply supported beam

$$v = C_1 \sin\left(\frac{n\pi x}{L}\right)$$

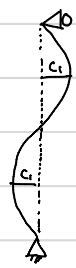
mode shape $\quad n=1$ (1st mode)

$n=2$ (2nd mode)

* $P_{cr} = \dfrac{\pi^2 EI}{L^2}$ *



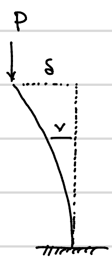
mode 1



mode 2

$P_{cr} = \dfrac{4\pi^2 EI}{L^2}$

What about other support (boundary) conditions?



Equilibrium $\quad M = P(\delta - v) \quad \therefore \quad EIv'' = P(\delta - v)$

$v'' + k^2 v = \underline{\underline{k^2 \delta}}$ $\quad$ nonzero RHS

solution = homogeneous solution (RHS = 0)

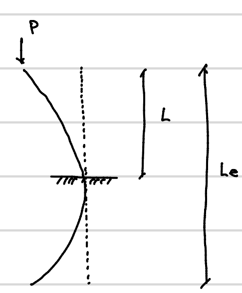
+ particular solution (RHS = $k^2\delta$)

$P_{cr} = \dfrac{\pi EI}{4L^2}$

In general: $\quad P_{cr} = \dfrac{\pi^2 EI}{(KL)^2}$ $\quad L_e = KL$ "effective length"

simply supported: $\quad L_e = 1.0\,L$

cantilevered: $\quad L_e = 2.0\,L$



* Effective length is the distance *

between inflection points ($M=0$)!

pinned-pinned	fixed-free	fixed-fixed	fixed-pinned
$K = 1.0$	$K = 2.0$	$K = 0.5$	$K = 0.7$
$P_{cr} = \dfrac{\pi^2 EI}{L^2}$	$P_{cr} = \dfrac{\pi^2 EI}{4L^2}$	$P_{cr} = \dfrac{4\pi^2 EI}{L^2}$	$P_{cr} = \dfrac{100\pi^2 EI}{49L^2}$

(truss $K = 1.0$)

<u>Critical Stress</u>

(pin-pin)

$\sigma_{cr} = \dfrac{P_{cr}}{A} = \dfrac{\pi^2 EI}{AL^2}$ radius of gyration $r = \sqrt{I/A}$

$\sigma_{cr} = \dfrac{\pi^2 E}{(L/r)^2}$ where $\dfrac{L}{r} \equiv$ slenderness ratio (typ. 30 - 150)

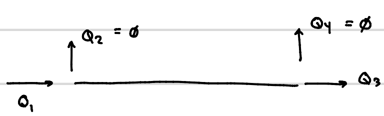
takeaway

check strength vs. applied stress *

for compression : also check for buckling *

tension - stabilizing

compression - de-stabilizing



Q_3 (+) tension

<u>Q_3 (-)</u> compression

$P_{cr} = \dfrac{\pi^2 EI}{L^2}$

$|Q_3| < P_{cr}$ no buckling

$|Q_3| \geq P_{cr}$ buckling