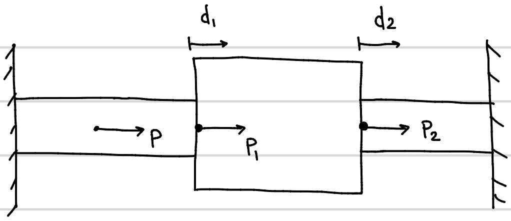
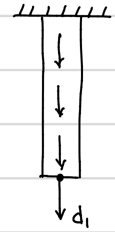
February 12, 2026
Matrix Displacement Method
Uniaxial Structures
Part IV - Fixed End Forces

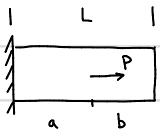


Matrix Displacement Method (MDM) "discretizes" structures into elements with loads applied at degrees of freedom (nodes, n) where we solve linear system of n # equations: $\{P\} = [S]\{d\}$

How to handle <u>interior</u> loads NOT applied at DOFs ?

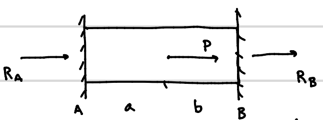
e.g. self-weight





E,A

Concentrated interior force



$\Delta = 0$

‡ Fixed End Forces ‡

Equivalent nodal loading is determined from technique known by Fixed-end-forces (FEF).

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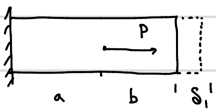
* Use force superposition * (as opposed to prior displacement superposition)



Remove redundant reaction (apply interior loading)

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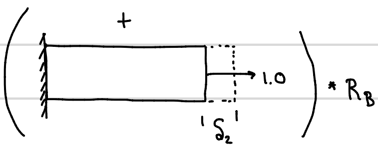


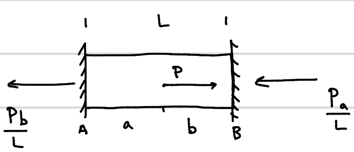
Apply scaled unit reaction

Displacement compatibility: $\Delta = \delta_1 + \delta_2 = 0$ $\qquad \delta = \dfrac{PL}{EA} \qquad \delta_1 = \dfrac{Pa}{EA} \qquad \delta_2 = \left(\dfrac{1.0\,L}{EA}\right) * R_B$

(before, force equilibrium) $\qquad \dfrac{Pa}{EA} + \dfrac{R_B L}{EA} = 0 \qquad \therefore \; R_B = -\dfrac{Pa}{L}$

Using overall force equilibrium: $\Sigma F_x = 0 \qquad R_A + R_B + P = 0 \qquad R_A - \dfrac{Pa}{L} + P = 0$

$$R_A = \frac{Pa}{L} - P = \frac{Pa - PL}{L} = \frac{-P(L-a)}{L} = -\frac{Pb}{L}$$



* Member level *

$\{Q\} = [k]\{u\}$ w/o interior loads

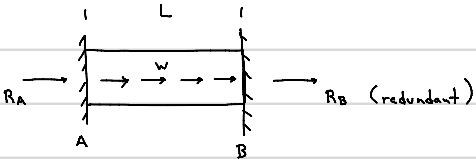
$\{Q - Q_f\} = [k]\{u\}$ w/ interior loading

member-end forces

fixed end forces (reactions) from interior loads

(negative sign = opposite direction of reactions)

Uniformly distributed axial load



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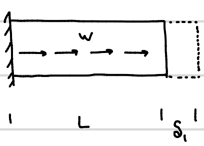
$\Delta = 0$



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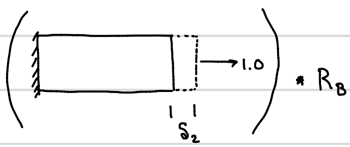
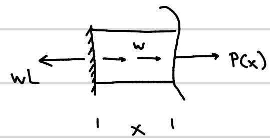


$$\delta(x) = \int \frac{P(x)\,dx}{EA}$$



$$\Sigma F_x = 0 \qquad P(x) + wx - wL = 0$$

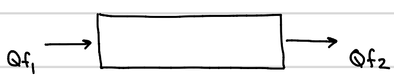
$$P(x) = wL - wx$$

$$\delta(x) = \int \frac{w(L-x)}{EA}\,dx = \frac{w}{EA}\int_0^L (L-x)\,dx = \frac{w}{EA}\Big|_0^L Lx - \frac{x^2}{2} = \boxed{\frac{wL^2}{2EA}}$$

δ_1

$$\Delta = \delta_1 + \delta_2 = 0 \qquad \delta_2 = \frac{R_B L}{EA} \qquad \frac{wL^2}{2EA} + \frac{R_B L}{EA} = 0$$

$$R_B = \frac{-wL}{2} \quad \therefore \quad R_A = \frac{-wL}{2}$$



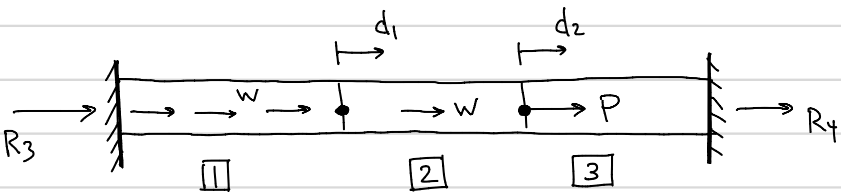
$$\{Q_f\} = \begin{Bmatrix} R_A \\ R_B \end{Bmatrix} = \begin{Bmatrix} -wL/2 \\ -wL/2 \end{Bmatrix}$$

Member-level $\{Q - Q_f\} = [k]\{u\}$

Structural-level $\{P - P_f\} = [S]\{d\}$

P: external force vector of loads applied at DOFs

P_f: assembled fixed-end-force vector from member level Qf contributions



$$\{P - P_f\} = [S]\{d\}$$

$$\downarrow$$

$$\begin{Bmatrix} 0 \\ P \end{Bmatrix}$$ force vector applied at DOFs

	u_1, Q_1	u_2, Q_2
1	3	1
2	1	2
3	2	4

$$[S] = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & k_{22}^1 \;\ldots & \ldots \\ 2 & \ldots & \ldots \end{array}$$

$$* \{P_f\} = \begin{array}{c} 1 \\ 2 \end{array} \begin{Bmatrix} Q_{f_2}^1 + Q_{f_1}^2 \\ Q_{f_2}^2 \end{Bmatrix} *$$

$$[k]^1 = \begin{array}{c} \\ 3 \\ 1 \end{array} \begin{array}{cc} 3 & 1 \\ \begin{bmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{bmatrix} \end{array}$$

$$\{Q_f\}^1 = \begin{array}{c} 3 \\ 1 \end{array} \begin{Bmatrix} Q_{f_1}^1 \\ Q_{f_2}^1 \end{Bmatrix} \qquad \{Q_f\}^2 = \begin{array}{c} 1 \\ 2 \end{array} \begin{Bmatrix} Q_{f_1}^2 \\ Q_{f_2}^2 \end{Bmatrix} \qquad \{Q_f\}^3 = \{\emptyset\}$$