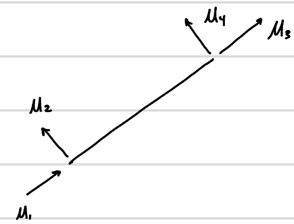
March 12, 2026
Matrix Displacement Method
Beams (BEBT)
Part I - formulation, [k]

Uniaxial

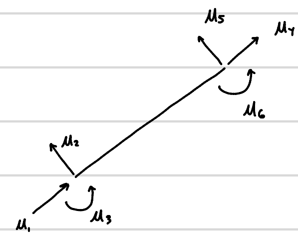
2D Truss



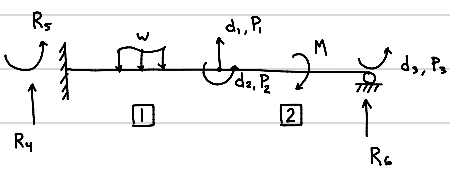
* **Beam** *

transverse displacements and rotations

2D Frame (later)



axial / transverse displacements and rotations



Structural-level Superposition

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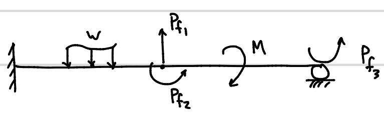
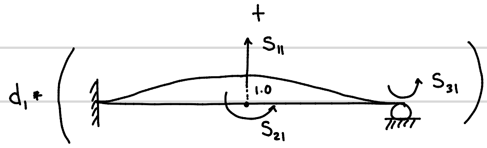


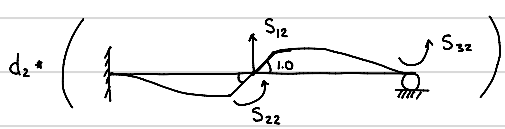
Apply loading $\quad d_1 = d_2 = d_3 = 0$

\+



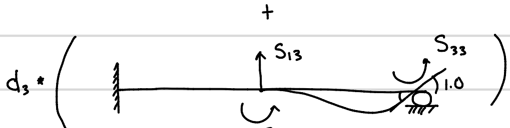
$d_1 = 1.0, \; d_2 = d_3 = 0$

\+



$d_2 = 1.0, \; d_1 = d_3 = 0$

\+



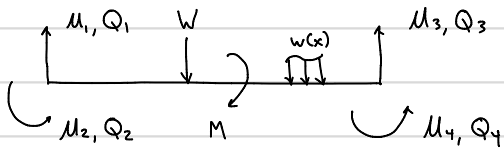
$d_3 = 1.0, \; d_1 = d_2 = 0$

$$P_1 = P_{f_1} + S_{11} d_1 + S_{12} d_2 + S_{13} d_3$$
$$P_2 = P_{f_2} + S_{21} d_1 + S_{22} d_2 + S_{23} d_3$$
$$P_3 = P_{f_3} + S_{31} d_1 + S_{32} d_2 + S_{33} d_3$$

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} P_{f_1} \\ P_{f_2} \\ P_{f_3} \end{Bmatrix} + \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

$$\{P - P_f\} = [S]\{d\}$$

No difference in local/global coordinates for beams $\{Q\}, [k], \{u\}$



Member-level superposition

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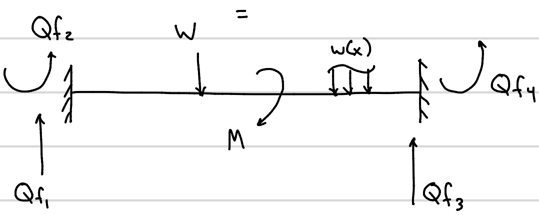
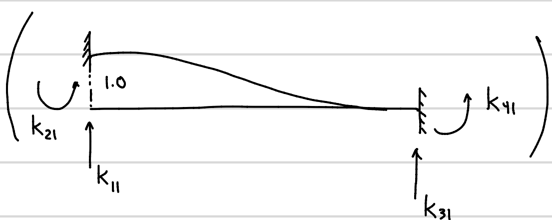


Apply loading $u_1 = u_2 = u_3 = u_4 = 0$

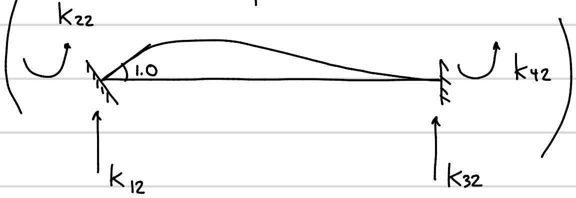
+



$* \, u_1$

$u_1 = 1.0, \; u_2 = u_3 = u_4 = 0$

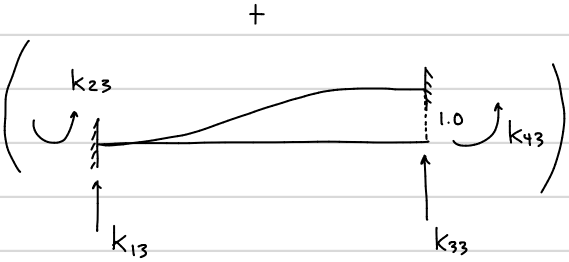
+



$* \, u_2$

$u_2 = 1.0, \; u_1 = u_3 = u_4 = 0$

+



$* \, u_3$

$u_3 = 1.0, \; u_1 = u_2 = u_4 = 0$

+

$* \, u_4$

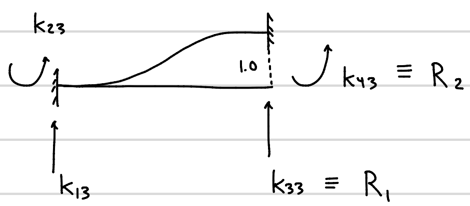
$u_4 = 1.0, \; u_1 = u_2 = u_3 = 0$

Now we write joint equilibrium equations

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} Qf_1 \\ Qf_2 \\ Qf_3 \\ Qf_4 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \qquad \{Q\} = \{Qf\} + [k]\{u\}$$

Need to derive stiffness terms

<u>3</u>rd column of $[k]$, $u_{\underline{3}} = 1.0$, $u_1 = u_2 = u_4 = 0$



indeterminate (2nd degree) – use superposition !

2 redundants (flexibility approach)

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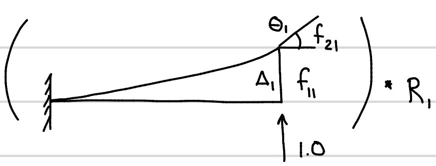

$R_1 = 1.0,\ R_2 = 0$

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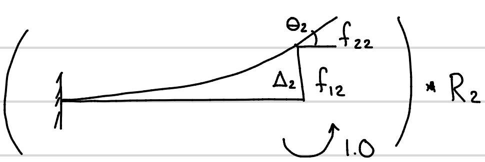


$R_1 = 0\ ,\ R_2 = 1.0$

<u>Axial</u> $P = \left(\frac{EA}{L}\right)\delta$ ← displacement; $\frac{EA}{L}$ ← stiffness (k)

$\delta = \left(\frac{L}{EA}\right)P$

$\frac{L}{EA}$ ← flexibility $\frac{1}{k} \equiv f$

<u>Compatibility</u>

$u_3 = \Delta_1 + \Delta_2 = 1.0 \qquad 1.0 = f_{11}R_1 + f_{12}R_2$

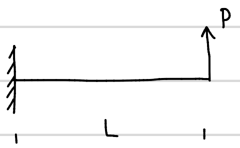
$u_4 = \theta_1 + \theta_2 = 0 \qquad 0 = f_{21}R_1 + f_{22}R_2$

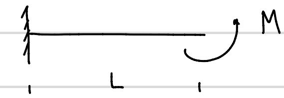
$$\begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

* Stiffness-based superposition : apply unit translations/rotations ⟶ equations from joint equilibrium

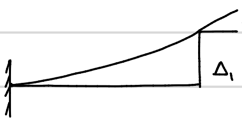
Flexibility-based superposition : apply unit forces/moments ⟶ equations from joint compatibility

We already know how to derive flexibilities (from moment-curvature)





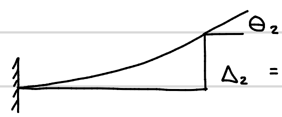
E,I



$\theta_1 = \frac{PL^2}{2EI}$

$\Delta_1 = \frac{PL^3}{3EI}$

E,I



$\theta_2 = \frac{ML}{EI}$

$\Delta_2 = \frac{ML^2}{2EI}$

$\Delta_1 = f_{11} P \qquad f_{11} = \frac{L^3}{3EI}$

$\theta_1 = f_{21} P \qquad f_{21} = \frac{L^2}{2EI}$

$\Delta_2 = f_{12} M \qquad f_{12} = \frac{L^2}{2EI}$

$\theta_2 = f_{22} M \qquad f_{22} = \frac{L}{EI}$

$$\begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \qquad [f] = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$

Solving system of equations yields $R_1 = \frac{12EI}{L^3} = k_{33} \qquad R_2 = \frac{-6EI}{L^2} = k_{43}$

From element equilibrium $\Sigma F_y = 0 \quad k_{13} + k_{33} = 0 \quad \therefore \quad k_{13} = \frac{-12EI}{L^3}$

$\Sigma M_b = 0 \quad k_{23} + k_{43} + k_{33} \cdot L = 0 \quad \therefore \quad k_{23} = \frac{-6EI}{L^2}$

Following the same superposition procedure for the 1st, 2nd, and 4th DOFs (i.e. columns)

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$