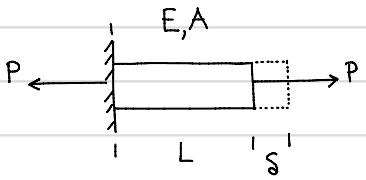
February 03, 2026
Matrix Displacement Method
Uniaxial Structures
Part I - Intro

Uniaxial Structures



Equilibrium: $\sigma = P/A$

Constitutive: $\sigma = E\varepsilon$

Kinematics: $\varepsilon = \delta/L$

$$P/A = E\varepsilon = E\,\delta/L \qquad \delta = \frac{PL}{EA}$$

L = original length

E = elastic modulus

A = cross-sectional area

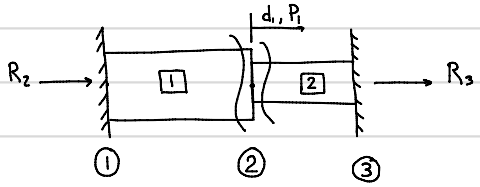
δ = displacement

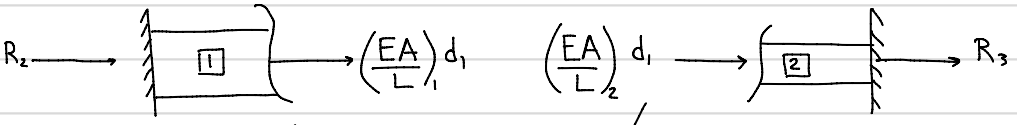
$$P = \left(\frac{EA}{L}\right)\delta$$

$\frac{EA}{L}$: stiffness; P: force; δ: displacement ↓ degree of freedom (DOF)

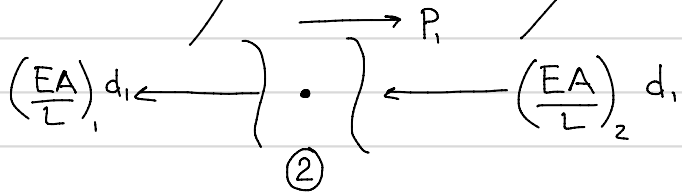
Spring: $F = kx$

1-DOF System





Newton's 3rd Law!



Notation

d = DOF

P = force

R = reaction

[#] = member

(#) = joint

Joint ② Equilibrium

$$\sum F_x = 0 \qquad P_1 - \left(\frac{EA}{L}\right)_1 d_1 - \left(\frac{EA}{L}\right)_2 d_1 = 0$$

$$P_1 = \left(\frac{EA}{L}_1 + \frac{EA}{L}_2\right) d_1$$

Structural-level relationship

$$P_1 = S_{11}\, d_1$$

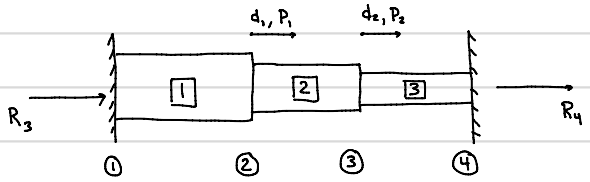
P_1: externally applied load; d_1: DOF

S is structural stiffness comprised of member-level contributions

S_{ij} = stiffness at DOF i associated with a unit displacement at DOF j

Matrix Displacement Method (MDM)

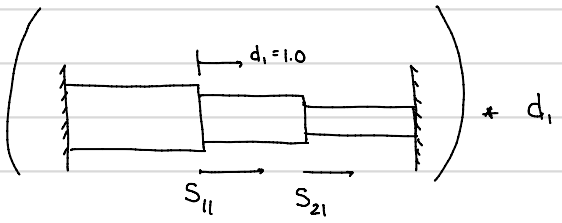
2-DOF



Superposition

* Structural Level *

=



Note: Force at DOF i = stiffness × displacement

$d_1 = 1 \quad d_2 = 0$

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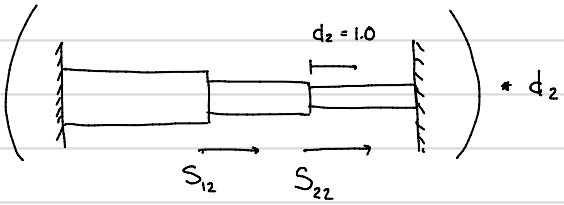


$d_1 = 0 \quad d_2 = 1$

Force Equilibrium

$$\{P\} = [S]\{d\}$$

Joint ② $P_1 = S_{11} d_1 + S_{12} d_2$

Joint ③ $P_2 = S_{21} d_1 + S_{22} d_2$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\{d\} = [S]^{-1}\{P\}$$

n # of dofs

$\{P\} = n \times 1$

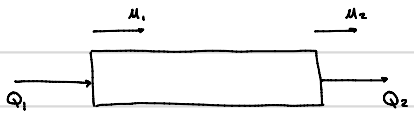
$[S] = n \times n$ $\quad \underset{n\times 1}{\{P\}} = \underset{n\times n}{[S]}\underset{n\times 1}{\{d\}}$ $\quad [S]$ - symmetric, i.e. $S_{12} = S_{21}$

$\{d\} = n \times 1$

Member Level

$[S]$ is comprised of member stiffness contributions

generic uniaxial member 2 DOF

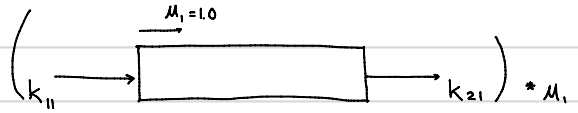


Q_i = force at DOF i

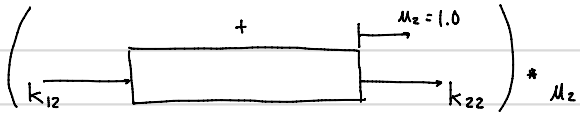
u_i = displacement at DOF i

k_{ij} = stiffness at DOF i associated w/ unit displacement at DOF j

$u_1 = 1 \quad u_2 = 0$



$u_1 = 0 \quad u_2 = 1$



Force equilibrium at DOFs

$$Q_1 = k_{11} u_1 + k_{12} u_2$$

$$Q_2 = k_{21} u_1 + k_{22} u_2$$

$$\{Q\} = [k]\{u\}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

what are the k_{ij}s ?