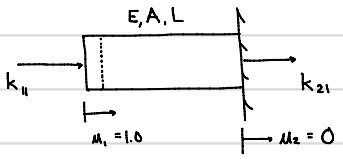
February 05, 2026
Matrix Displacement Method
Uniaxial Structures
Part II - [k], assembly of [S]



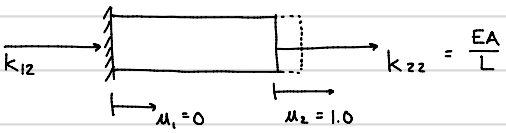
$F = \left(\frac{EA}{L}\right)\delta$ (force = stiffness × displacement)

$F = k_{11}\,\delta\,(=1.0)$

$k_{11} = \frac{EA}{L}$

$\Sigma F_x = 0 \quad k_{11} + k_{21} = 0$

$\therefore k_{21} = -\frac{EA}{L}$



$\Sigma F_x = 0 \quad k_{12} + k_{22} = 0$

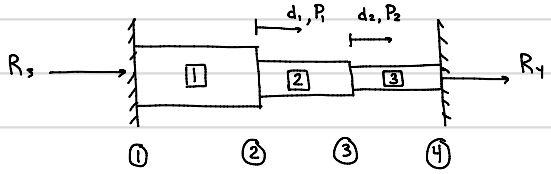
$k_{12} = -\frac{EA}{L}$

$$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

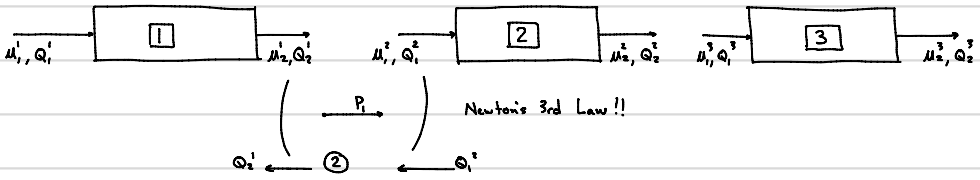
symmetric

How do the k_{ij}s assemble into S_{ij}

member-level — structural-level



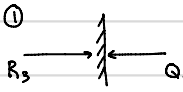
$\{P\} = [S]\{d\}$



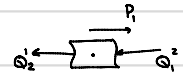
Newton's 3rd Law !!

* Rigorous assembly via joint equilibrium

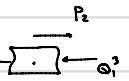
Joint ①



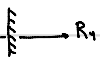
$\Sigma F_x = 0 \quad R_3 = Q_1'$

Joint ② 

$\Sigma F_x = 0 \quad P_1 = Q_2' + Q_1^2$ * at DOF 1

Joint ③ 

$\Sigma F_x = 0 \quad P_2 = Q_2^2 + Q_1^3$ * at DOF 2

Joint ④ 

$\Sigma F_x = 0 \quad R_4 = Q_2^3$

at DOF 1 $P_1 = Q_2^1 + Q_1^2 = \left(k_{21}^1 u_1^1 + k_{22}^1 u_2^1\right) + \left(k_{11}^2 u_1^2 + k_{12}^2 u_2^2\right)$

at DOF 2 $P_2 = Q_2^2 + Q_1^3 = \left(k_{21}^2 u_1^2 + k_{22}^2 u_2^2\right) + \left(k_{11}^3 u_1^3 + k_{12}^3 u_2^3\right)$

Recall $\{Q\} = [k]\{u\}$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Compatibility

at joint ② : $d_1 = u_2^1 = u_1^2$ $\quad P_1 = \left(k_{21}^1 \cancel{u_1^1} + k_{22}^1 \cancel{u_2^1}\right) + \left(k_{11}^2 \cancel{u_1^2} + k_{12}^2 \cancel{u_2^2}\right)$ (u_1^1: support/B.C.s = 0; $u_1^2 = u_2^1 = d_1$; $u_2^2 = d_2$)

at joint ③ : $d_2 = u_2^2 = u_1^3$ $\quad P_2 = \left(k_{21}^2 \cancel{u_1^2} + k_{22}^2 \cancel{u_2^2}\right) + \left(k_{11}^3 \cancel{u_1^3} + k_{12}^3 \cancel{u_2^3}\right)$ ($u_1^3 = u_2^2 = d_2$; u_2^3: support/B.C.s = 0)

$$\{P\} = [S]\{d\} \qquad \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} k_{22}^1 + k_{11}^2 & k_{12}^2 \\ k_{21}^2 & k_{22}^2 + k_{11}^3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$\downarrow$

$[S]$

symmetric since $k_{12}^2 = k_{21}^2$

Assembly

$$\{P\} = [S]\{d\}$$

$\downarrow$

combination of member k_{ij}s

Solution

$$\{d\} = [S]^{-1}\{P\}$$

compatibility $d - u$

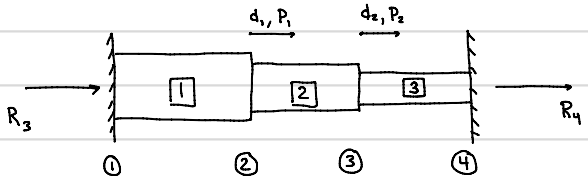
Post-processing

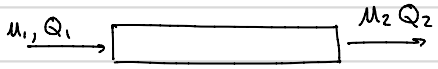
$$\{Q\} = [k]\{u\}$$

can solve for $\{Q\}_s$ $\longrightarrow$ stresses $\sigma = \frac{\text{force}}{\text{area}}$

$\longrightarrow$ reactions (joint equilibrium)

Code # Assembly





member # \ code #	u_1, Q_1	u_2, Q_2
1	3	1
2	1	2
3	2	4

code # is # associated with structural DOF/reaction

1 $[k]^1 = \begin{matrix} & 3 & 1 \\ 3 & k_{11}^1 & k_{12}^1 \\ 1 & k_{21}^1 & k_{22}^1 \end{matrix}$

2 $[k]^2 = \begin{matrix} & 1 & 2 \\ 1 & k_{11}^2 & k_{12}^2 \\ 2 & k_{21}^2 & k_{22}^2 \end{matrix}$

3 $[k]^3 = \begin{matrix} & 2 & 4 \\ 2 & k_{11}^3 & k_{12}^3 \\ 4 & k_{21}^3 & k_{22}^3 \end{matrix}$

We are concerned with structural DOFs 1, 2

↓

only select k_{ij}s with code #'s 1, 2

$$[S] = \begin{matrix} & 1 & 2 \\ 1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 2 & k_{21}^2 & k_{22}^2 + k_{11}^3 \end{matrix}$$

same [S] from rigorous assembly!