

## Linear Algebra Fundamentals

If I gave you the three (linear) equations:

$$\begin{aligned}3x_1 + 2x_2 + 1x_3 &= 10 \\2x_1 + 4x_2 + 3x_3 &= 19 \\1x_1 + 3x_2 + 5x_3 &= 22\end{aligned}$$

Could you solve these? How?

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 19 \\ 22 \end{Bmatrix} \qquad [A][x] = [B] \qquad [A]\{x\} = \{B\}$$

$$[A]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots \\ \dots & \dots & a_{ij} & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad m \times n = \text{rows by columns}$$

In the preceding case: [A] is a 3 x 3 “square matrix”  
[x] is a 3 x 1 “column matrix/vector”  
[B] is a 3 x 1 “column matrix/vector”

$$[A]_{2 \times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad [B]_{3 \times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ square matrix} \qquad \begin{bmatrix} a_{11} & \dots & \dots \\ \dots & a_{22} & \dots \\ \dots & \dots & a_{33} \end{bmatrix} \text{ main diagonal}$$

**Transpose** of a matrix:  $[A_{ij}]^T = [A]_{ji}$        $[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

**Symmetric Matrix:**  $A = A^T$ , i.e.  $a_{ij} = a_{ji}$        $[A] = \begin{bmatrix} 1 & 4 & -5 \\ 4 & 2 & 6 \\ -5 & 6 & 3 \end{bmatrix}$

**Diagonal Matrix:**  $a_{ij} = 0$  for  $i \neq j$        $[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**Identity Matrix:**  $a_{ij} = 1$  for  $i = j$ ,  $a_{ij} = 0$  for  $i \neq j$        $[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Null Matrix:**  $a_{ij} = 0$        $[O] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Equality**  $[A] = [B]$ :      If two matrices are of the same order (dimensions) and all of their elements are identical  $a_{ij} = b_{ij}$

### Matrix Operations

**Addition and Subtraction:** Only matrices of the same order (conformable) can be added/subtracted by operating on corresponding elements:  $[A] + [B] = a_{ij} + b_{ij}$

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad [B] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad [A+B] = \begin{bmatrix} 2 & 2 & 4 \\ 4 & 6 & 6 \end{bmatrix}$$

$$[A-B] = \begin{bmatrix} 0 & 2 & 2 \\ 4 & 4 & 6 \end{bmatrix} \quad [B-A] = \begin{bmatrix} 0 & -2 & -2 \\ -4 & -4 & -6 \end{bmatrix}$$

**Multiplication by a scalar:** To multiply by a scalar, each element of the matrix must be multiplied by the scalar:  $s[A] = s(a_{ij})$

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad s=2 \quad s[A] = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

**Multiplication of matrices:** This can only be carried out if the number of columns of the first matrix [A] matches the number of rows in the second matrix [B], where the resulting matrix [C] has dimensions equal to the number of rows of the first matrix by the number of columns of the second matrix:  $[A]_{m \times n} [B]_{n \times p} = [C]_{m \times p}$

This can be carried out by algebraically summing each element of the  $i$ th row of [A]

by the corresponding element of the  $j$ th row of [B].  $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad [B] = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \quad [C] = [A][B]$$

$$[C] = \begin{pmatrix} (1 \times 1 + 2 \times 2 + 3 \times 3) & (1 \times 4 + 2 \times 5 + 3 \times 6) & (1 \times 7 + 2 \times 8 + 3 \times 9) \\ (4 \times 1 + 5 \times 2 + 6 \times 3) & (4 \times 4 + 5 \times 5 + 6 \times 6) & (4 \times 7 + 5 \times 8 + 6 \times 9) \end{pmatrix}$$

$$[C] = \begin{bmatrix} 14 & 32 & 50 \\ 32 & 77 & 122 \end{bmatrix} \quad \text{**Note: } [A] \text{ pre-multiplies } [B]; [B][A] \text{ not possible!}$$

Even when  $[A][B]$  and  $[B][A]$  is possible, generally:  $[A][B] \neq [B][A]$

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, [B] = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \quad [A][B] = \begin{bmatrix} 11 & 5 \\ 25 & 11 \end{bmatrix} \quad [B][A] = \begin{bmatrix} 6 & 10 \\ 10 & 16 \end{bmatrix}$$

It is therefore important to maintain the proper sequential order of matrices when computing matrix products.

**Some common relations:**

$$[A]([B]+[C])=[A][B]+[A][C]$$

$$([A][B])^T=[B]^T[A]^T \quad ([A][B][C])^T=[C]^T[B]^T[A]^T$$

Multiplication of any matrix  $[A]$  by a conformable (same dimensions) **null** matrix  $[O]$  yields a null matrix:

$$[A][O]=[O] \quad [O][A]=[O]$$

Multiplication of any matrix  $[A]$  by any conformable (same dimensions) **identity** matrix  $[I]$  yields the original matrix:

$$[A][I]=[A] \quad [I][A]=[A]$$

**Inverse of a square matrix:** The inverse is only defined for square matrices  $[A]$  as a matrix  $[A]^{-1}$  where pre-multiplication of the original matrix by the inverse yields the identity matrix  $[I]$ .

$$[A]^{-1}[A]=[I]$$

Thus for a system of linear equations as initially described  $[A][x]=[B]$

The concept of an inverse is used to solve for the unknown variables:

$$[A]^{-1}[A][x]=[A]^{-1}[B] \quad [I][x]=[A]^{-1}[B] \quad [x]=[A]^{-1}[B]$$

In general, inverting a square matrix is computationally expensive and thus more economical solution techniques are employed for solving linear (matrix) systems of equations such as LU (lower-upper) factorization.

**Orthogonality**  $[Q]^{-1}=[Q]^T$   $[Q]=\begin{bmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$